Tikrit university
College of Engineering
Mechanical Engineering Department

Lectures on Numerical Analysis

Chapter 6
Numerical Interpolation

Assistant prof. Dr. Eng. Ibrahim Thamer Nazzal

Introduction to Interpolation

Interpolation is **a method of deriving a simple function from the given discrete data set such that the function passes through the provided data points**. This helps to determine the data points in between the given data ones.

Interpolation function: a function that passes exactly through a set of data points.

- In tables, the function is only specified at a limited number.
- We can use interpolation to find functional values at other values of the independent variable, e.g. sin(0.63253)

x	sin(x)
0.0	0.000000
0.5	0.479426
1.0	0.841471
1.5	0.997495
2.0	0.909297
2.5	0.598472

Many a times, continuous function $y = f(x)$ is given only at discrete points such as $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$

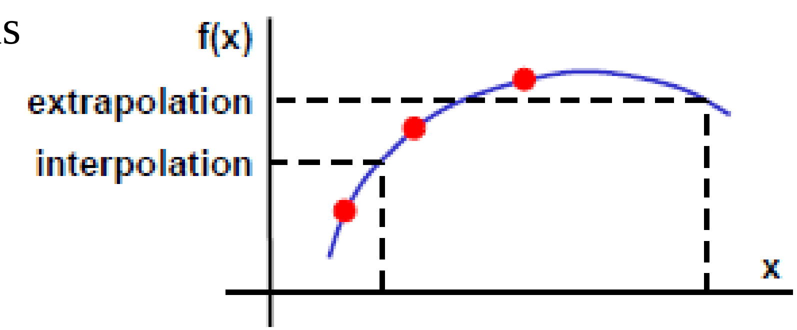
How does one find the value of y at any other value of x?

Then one can find the value of y at any other value of x.
This is called interpolation.
Of course, if x falls outside the range of x for which the data is given, it is no longer interpolation but instead is called **extrapolation.**

Polynomials are the most common choice of interpolants because they are easy to:

Evaluate, Differentiate, and Integrate

Polynomial Interpolation

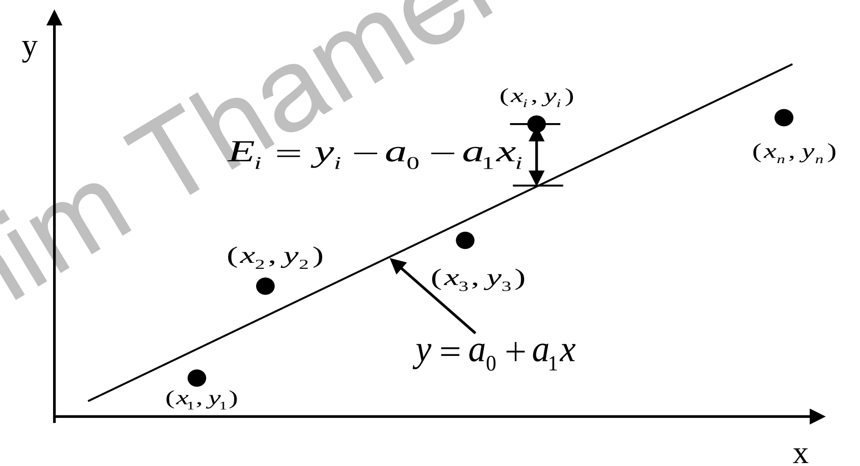


Direct Method

Given '*n+1*' data points (x_0,y_0), (x_1,y_1),………….. (x_n,y_n), pass a polynomial of order '*n*' through the data as given below:

$$y = a_0 + a_1 x + \ldots\ldots\ldots\ldots\ldots\ldots + a_n x^n.$$

where a_0, a_1,………………. a_n are real constants.



Polynomial Interpolation

- Newton's Divided Difference Interpolating Polynomials
- Lagrange Interpolating Polynomials

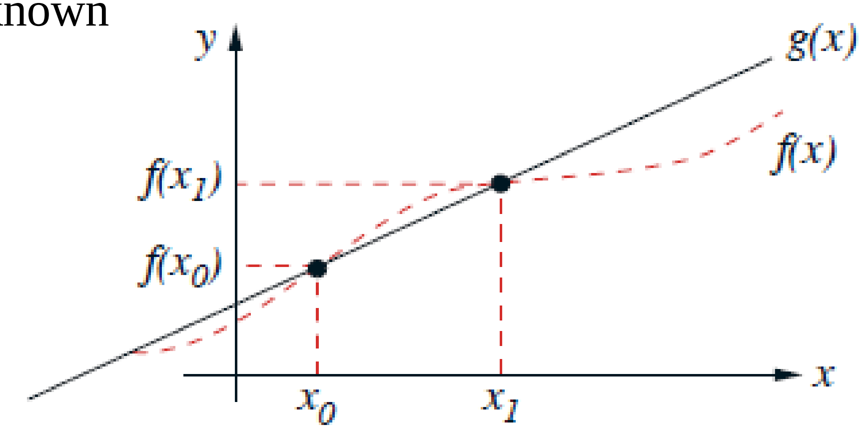
The question is to find the coefficients $a_0, a_1, \ldots, a_n$

Linear Interpolation

- **Linear interpolation is obtained by passing a straight line between 2 data points**

$f(x)$ = the exact function for which values are known only at a discrete set of data points.

$g(x)$ = the interpolated approximation to $f(x)$.



Given (x_0,y_0), (x_1,y_1), …… (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.

Given *n* data points

best fit $y = a_0 + a_1 x$ to the data.

Example 1

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at t=16 seconds using the direct method for linear interpolation.

Solution

Table 1 Velocity as a function of time.

$t, (\text{s})$	$v(t), (\text{m/s})$
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Linear Interpolation

$$v(t) = a_0 + a_1 t$$

$$v(15) = a_0 + a_1(15) = 362.78$$

$$v(20) = a_0 + a_1(20) = 517.35$$

Solving the above two equations gives,

$$a_0 = -100.93$$

$$a_1 = 30.914$$

Hence

$$v(t) = -100.93 + 30.914t, \ 15 \le t \le 20.$$

$$v(16) = -100.93 + 30.914(16) = 393.7 \text{ m/s}$$

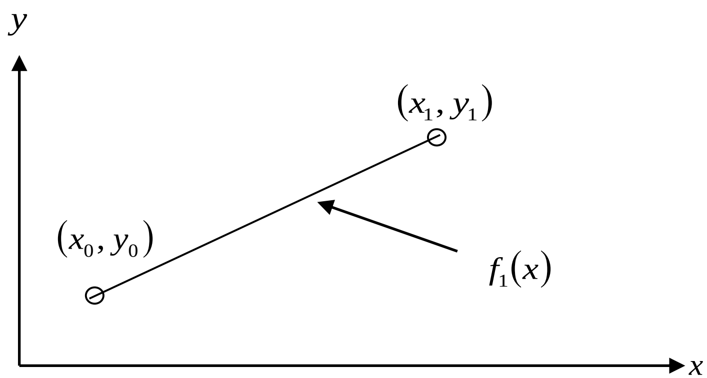


Figure 3 Linear interpolation.

Example 2

The upward velocity of a rocket is given as a function of time in Table 2. Find the velocity at t=16 seconds using the direct method for quadratic interpolation.

Table 2 Velocity as a function of time.

t, (s)	$v(t)$, (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Solution

$$v(t) = a_0 + a_1 t + a_2 t^2$$

$$v(10) = a_0 + a_1(10) + a_2(10)^2 = 227.04$$

$$v(15) = a_0 + a_1(15) + a_2(15)^2 = 362.78$$

$$v(20) = a_0 + a_1(20) + a_2(20)^2 = 517.35$$

Solving the above three equations gives

$a_0 = 12.05$ $a_1 = 17.733$ $a_2 = 0.3766$

$$v(t) = 12.05 + 17.733t + 0.3766t^2, \quad 10 \le t \le 20$$

$$v(16) = 12.05 + 17.733(16) + 0.3766(16)^2 = 392.19 \text{ m/s}$$

The absolute relative approximate error obtained between the results from the first and second order polynomial is

$$|\in_a| = \left|\frac{392.19 - 393.70}{392.19}\right| \times 100$$

$$= 0.38410\%$$

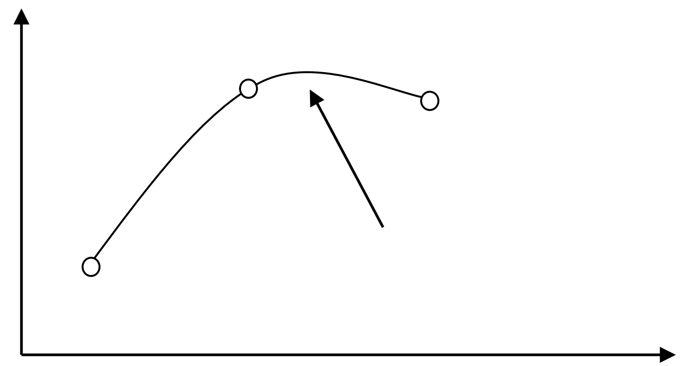


Figure Quadratic interpolation.

Linear interpolation: Given (x_0, y_0), pass a linear interpolant through the data (x_1, y_1),

Given any two points, $(x_0, f(x_0)), (x_1, f(x_1))$

- A straight line passes from these two points.

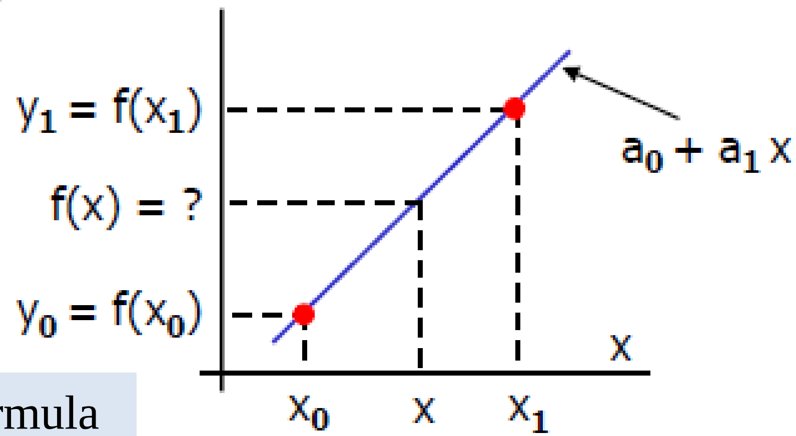
Using similar triangles

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

or

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$



$$f_1(x) = b_0 + b_1(x - x_0)$$ Linear interpolation formula

Where $$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Example 3

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at t=16 seconds using the Newton Divided Difference method for linear interpolation.

Table. Velocity as a function of time

t (s)	
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Solution

$$v(t) = b_0 + b_1(t - t_0)$$

$$t_0 = 15,\ v(t_0) = 362.78$$

$$t_1 = 20,\ v(t_1) = 517.35$$

$$b_0 = v(t_0) = 362.78$$

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0} = 30.914$$

$$v(t) = b_0 + b_1(t - t_0)$$

$$= 362.78 + 30.914(t - 15),\ 15 \le t \le 20$$

At $t = 16$

$$v(16) = 362.78 + 30.914(16 - 15)$$

$$= 393.69 \text{ m/s}$$

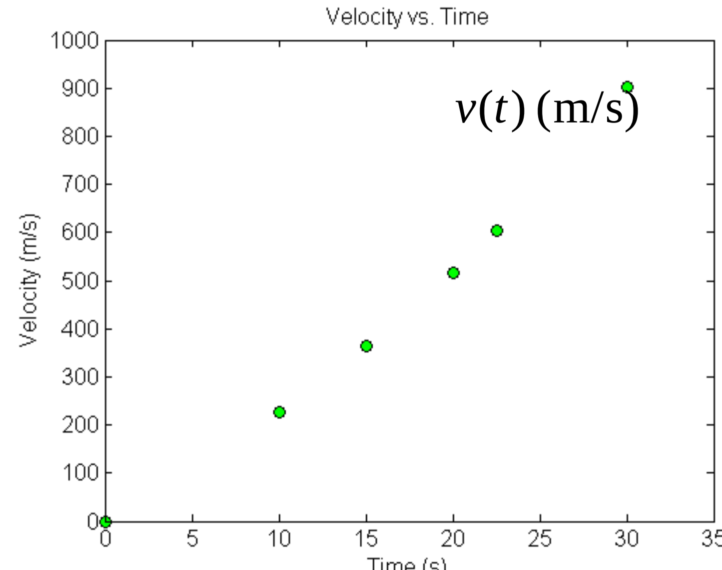


Figure. Velocity vs. time data for the rocket example

Quadratic Interpolation

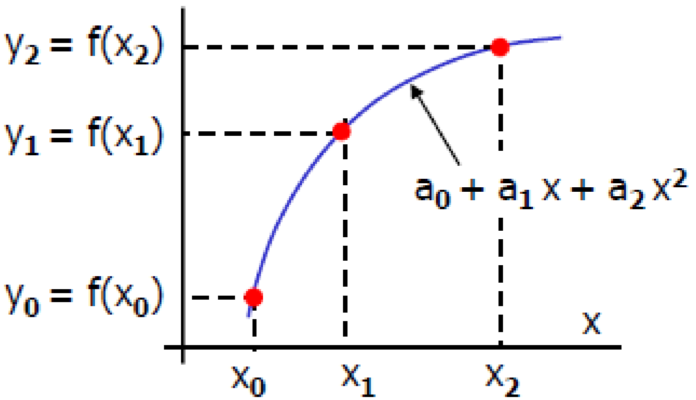
Given any three points: $(x_0, f(x_0)), (x_1, f(x_1)), and\ (x_2, f(x_2))$, fit quadratic interpolant through data

The polynomial that interpolates the three points is:

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$ **Quadratic interpolation formula**

- How to find b_0, b_1 and b_2 in terms of given quantities?
  - at $x = x_0$ $\quad f_2(x) = f(x_0) = b_0 \quad \rightarrow b_0 = f(x_0)$
- at $x = x_1$ $\quad f_2(x) = f(x_1) = b_0 + b_1 x_1 \quad \rightarrow b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$
- at $x = x_2$ $\quad f_2(x) = f(x_2) = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$

$$\rightarrow \mathbf{b}_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$



Example 4: The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at t=16 seconds using the Newton Divided Difference method for quadratic interpolation.

Table. Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Figure. Velocity vs. time data for the rocket example

$$t_0 = 10,\ v(t_0) = 227.04$$

$$t_1 = 15,\ v(t_1) = 362.78$$

$$t_2 = 20,\ v(t_2) = 517.35$$

$$b_0 = v(t_0) = 227.04$$

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0} = \frac{362.78 - 227.04}{15 - 10} = 27.148$$

$$b_2 = \frac{\dfrac{v(t_2) - v(t_1)}{t_2 - t_1} - \dfrac{v(t_1) - v(t_0)}{t_1 - t_0}}{t_2 - t_0} = \frac{\dfrac{517.35 - 362.78}{20 - 15} - \dfrac{362.78 - 227.04}{15 - 10}}{20 - 10}$$

$$= \frac{30.914 - 27.148}{10} = 0.37660$$

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1)$$

$$= 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15), \qquad 10 \le t \le 20$$

At $t = 16$,

$$v(16) = 227.04 + 27.148(16 - 10) + 0.37660(16 - 10)(16 - 15) = 392.19 \text{ m/s}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first order and second order polynomial is

$$|\in_a| = \left|\frac{392.19 - 393.69}{392.19}\right| x100$$

$$= 0.38502\ \%$$

General nth Order Interpolation

Given any n+1 points: $(x_0, f(x_0)), (x_1, f(x_1)), ..., (x_n, f(x_n))$

The polynomial that interpolates all points is:

$$f_n(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + ... + b_n(x - x_0)...(x - x_{n-1})$$

$$b_0 = f(x_0)$$

$$b_1 = f[x_0, x_1]$$

....

$$b_n = f[x_0, x_1, ..., x_n]$$

Newton's Forward Interpolation Formula

Let the function $y = f(x)$ take the values $y_0, y_1, \cdots, y_n$ corresponding to the values $x_0, x_1, \cdots, x_n$ of x. Let these values of x be equispaced such that $x_i = x_0 + ih (i = 0,1, \cdots)$. Assuming $y(x)$ to be a polynomial of the n^{th} degree in x such that $y(x_0) = y_0, y(x_1) = y_1, \cdots, y(x_n) = y_n$. We can write

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \cdots + a_n(x - x_0)(x - x_1) \cdots (x - x_{n-1}) \quad \ldots\ldots\ldots\ldots\ldots (1)$$

Putting $x = x_0, x_1, \cdots, x_n$ successively in (1), we get

$$y_0 = a_0, y_1 = a_0 + a_1(x_1 - x_0), y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

and *so* on.

From these, we find that $a_0 = y_0, \Delta y_0 = y_1 - y_0 = a_1(x_1 - x_0) = a_1 h$

$$\therefore\ a_1 = \frac{1}{h}\Delta y_0$$

Also $\Delta y_1 = y_2 - y_1 = a_1(x_2 - x_1) + a_2(x_2 - x_0)(x_2 - x_1)$

$$= a_1 h + a_2 hh = \Delta y_0 + 2h^2 a_2$$

$$\therefore\ a_2 = \frac{1}{2h^2}(\Delta y_1 - \Delta y_0) = \frac{1}{2!\,h^2}\Delta^2 y_0$$

Similarly $a_3 = \frac{1}{3!h^3}\Delta^3 y_0$ and so on.

Substituting these values in (1), we obtain

$$y(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!\,h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!\,h^3}(x - x_0)(x - x_1)(x - x_2) + \cdots$$

……………………………………………..(2)

Now if it is required to evaluate y for $x = x_0 + ph$, then $(x - x_0) = ph$,
$x - x_1 = x_0 + ph - (x_0 + h) = ph - h = (p-1)h$, and
$(x - x_2) = x_0 + ph - (x_0 + 2h) = (p-2)h$ etc.
Hence, writing $y(x) = y(x_0 + ph) = y_p$, (2) becomes

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \frac{p(p-1)\cdots(p-n-1)}{3!}\Delta^n y_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

It is **called Newton's forward interpolation formula** as (3) contains y_0 and the forward differences of y_0.

Otherwise: Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \cdots$ corresponding to the values $x_0, x_0 + h, x_0 + 2h, \cdots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_0 + ph$, where p is any real number.

Newton's Backward Interpolation Formula

Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots$ corresponding to the values $x_0, x_0 + h, x_0 + 2h, \cdots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_n + ph$, where p is any real number. Then we have

$$y_p = f(x_n + ph) = E^p f(x_n) = (1 - \nabla)^{-p} y_n \; [\because E^{-1} = 1 - \nabla]$$

$$= \left[1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_0 + \cdots\right] y_n$$

[using binomial theorem]

i.e., $y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$

It is called Newton's backward interpolation formula

EXAMPLE 5

The table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface:

x = height:	100	150	200	250	300	350	400
y = distance:	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of y when (i) $x = 160$ft (ii) $x = 410$.

Solution:

The difference table is as under:

x	y	Δ	Δ^2	Δ^3	Δ^4
100	10.63				
		2.40			
150	$\mathbf{13.03}$		-0.39		
		$\mathbf{2.01}$		0.15	
200	15.04		$\mathbf{-0.24}$		-0.07
		1.77		$\mathbf{0.08}$	
250	16.81		-0.16		$\mathbf{-0.05}$
		1.61		0.03	
300	18.42		-0.13		-0.01
		1.48		$\mathbf{0.02}$	
350	19.90		$\mathbf{-0.11}$		
		$\mathbf{1.37}$			
400	$\mathbf{21.27}$				

(i) If we take $x_0 = 150$, then $y_0 = 13.03, \Delta y_0 = 2.01, \Delta^2 y_0 = -0.24$, $\Delta^3 = 0.08, \Delta^4 y_0 = -0.05$

Since $x = 160$ and $h = 50$, $\therefore p = \frac{x - x_0}{h} = \frac{10}{50} = 0.2$

$\therefore$ Using Newton's forward interpolation formula, we get

$$y_{160} = y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots$$

$y_{160} = 13.03 + 0.402 + 0.192 + 0.0384 + 0.00168 = 13.46$ *nautical miles*

(ii) Since $x = 410$ is near the end of the table, we use Newton's backward interpolation formula.

$$\therefore \quad \textit{Taking}\; x_n = 400, p = \frac{x - x_n}{h} = \frac{10}{50} = 0.2$$

Using the line of backward difference

$$y_n = 21.27, \nabla y_n = 1.37, \nabla^2 y_n = -0.11, \nabla^3 y_n = 0.02 \; etc.$$

$\therefore$ Newton's backward formula gives

$$y_{410} = y_{400} + p\nabla y_{400} + \frac{p(p+1)}{2!}\nabla^2 y_{400} + \frac{p(p+1)(p+2)}{3!}\Delta^3 y_{400} + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_{400} + \cdots$$

$$= 21.27 + 0.2(1.37) + \frac{0.2(1.2)}{2!}(-0.11 + \frac{0.2(1.2)(2.2)}{3!}(0.02) + \frac{0.2(1.2)(2.2)(3.2)}{4!}(-0.01)$$

$$= 21.27 + 0.274 - 0.0132 + 0.0018 - 0.0007 \quad = 21.53 \; \textit{nautical miles}$$